Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Ideal Tail Load for Minimum Aircraft Drag

E. V. Laitone*
University of California, Berkeley, Calif.

Introduction

IT will be shown that aircraft can have their maximum weight to drag ratio (W/D) occur when the tail load is slightly positive (upward tail lift). This would correspond to the minimum trimmed drag for modern conventional aircraft with an aft-tail, except for the unfortunate fact that they do not have sufficient tail volume to permit the rearward c.g. location that would produce an upload on the tail and still maintain static longitudinal stability. Prior to W.W. II most aircraft were designed to cruise with either zero or slightly positive tail loads, but since then the design trend has been toward tail downloads at nearly all flight speeds. It will be shown that a typical transport aircraft could save up to 5% of its fuel consumption if the tail volume were increased so as to permit the aft c.g. location necessary for minimum trimmed drag.

Analysis

Munk's stagger theorem allows one to apply Prandtl's relation for the induced drag (D_i) of a biplane to a wing-tail combination[1-3]

$$\pi q D_i = \frac{L_1^2}{b_1^2} + 2\sigma \frac{L_1}{b_1}\frac{L_2}{b_2} + \frac{L_2^2}{b_2^2} \tag{1}$$

where L_1 is the lift force acting on the wing of span b_1, and L_2 is the lift force acting on the tail of span $b_2 \leq b_1$. The factor σ is the Prandtl coefficient[1-3] that is dependent upon the ratio $b_2/b_1 \leq 1$, and the "biplane gap" which is the vertical distance between the wing and the tail. σ represents the mutual downwash effects, and for elliptic lift distributions it is limited by[1-3]

$$\sigma \leq b_2/b_1 \leq 1 \tag{2}$$

When the wing and tail are in the plane of the freestream velocity vector so that the equivalent "biplane gap" is zero, we have $\sigma = b_2/b_1$ as used by Naylor[3] in his analysis. This proved that the minimum induced drag occurred when the tail lift (L_2) was zero, so that the induced drag became greater for $L_2 < 0$ (tail download) than it was for $L_2 = 0$. At first glance Eq. (1) apparently indicates the contrary, but Naylor's conclusions are correct because the increased induced drag produced by the wing and tail more than overcomes the so-called tail thrust given by the middle term of Eq. (1) when $L_2 < 0$; namely,

$$\frac{2\sigma}{\pi q}\frac{L_1}{b_1}\frac{L_2}{b_2} = qS_1\sigma\left(\frac{b_1}{b_2}\right)\left(\frac{2C_{L_1}}{\pi A_1}\right)\left(C_{L_2}\frac{S_2}{S_1}\right) \tag{3}$$

where $L_1 = qS_1C_{L_1}$, $L_2 = qS_2C_{L_2}$, and the wing aspect ratio is given by $A_1 = b_1^2/S_1$, where S_1 is its planform area.

We will now generalize Naylor's results to show that the minimum induced drag occurs with a tail upload, if the tail is either above or below the wing, because in these cases $\sigma < b_2/b_1$. We eliminate the wing lift by introducing the gross weight (W):

$$L_1 = W - L_2 = C_{L0}qS_1 - C_{L_2}qS_2 = C_{L_1}qS_1 \tag{4}$$

so Eq. (1) becomes

$$\pi q D_i = \frac{W^2}{b_1^2} - 2\left(1 - \pi\frac{b_1}{b_2}\right)\left(\frac{WL_2}{b_1^2}\right) + \left(1 - 2\sigma\frac{b_1}{b_2} + \frac{b_1^2}{b_2^2}\right)\left(\frac{L_2}{b_1}\right)^2 \tag{5}$$

Note that, for $\sigma < b_2/b_1$, the middle term proves that $L_2 < 0$ produces a greater induced drag than does $L_2 > 0$ (tail upload). The minimum induced drag occurs when

$$\frac{L_2}{W} = \left(1 - \sigma\frac{b_1}{b_2}\right)\left(1 - 2\sigma\frac{b_1}{b_2} + \frac{b_1^2}{b_2^2}\right)^{-1} > 0 \tag{6}$$

For example, if the tail is above the wing at a height given by $0.1\ b_1$, when $b_2/b_1 = 0.3$, then[2] $\sigma = 0.235$ and Eq. (6) gives $L_2/W = 0.0209$. This is a tail upload that cannot be achieved by most modern transport aircraft.

Now we will show that an even greater tail upload is required for minimum aircraft drag when the usual aerodynamic interference between the fuselage, wing, and tail are included. For this consideration we introduce the following for the aircraft's total drag coefficient:

$$C_D = C_{De} + \frac{C_{L_1}^2}{\pi A_1 e_1} + \sigma\left(\frac{b_1}{b_2}\right)\left(\frac{2CL_1}{\pi A_1}\right)\left(C_{L_2}\frac{S_2}{S_1}\right) + \frac{C_{L_2}^2}{\pi A_2 e_2}\left(\frac{S_2}{S_1}\right) = \frac{D_e + D_i}{qS_1} \tag{7}$$

All of the aircraft interference drag acting on the wing ($e_1 < 1$) and on the tail ($e_2 < 1$) is included in the second and fourth terms only, since the third term could only be decreased by any variation from the ideal elliptic lift distribution or any departure from ideal potential flow. The first term includes all of the aircraft's frictional and compressibility drag effects that can be considered approximately independent of C_{L_1} and C_{L_2} in the range considered. Then, if we introduce Eq. (4) into Eq. (7) we obtain

$$C_D = C_{De} + (C_{L0}^2/\pi A_1 e_1) - K_1 C_{L_2} + K_2 C_{L_2}^2 \tag{8}$$

where

$$K_1 = \left(1 - e_1\sigma\frac{b_1}{b_2}\right)\left(\frac{2C_{L0}}{\pi A_1 e_1}\right)\left(\frac{S_2}{S_1}\right) > 0 \tag{9}$$

Received Sept. 13, 1977; revision received Dec. 19, 1977.

Index categories: Performance; Configuration Design.

*Professor, Dept. of Mechanical Engineering. Associate Fellow AIAA.

$$K_2=\left[1+\frac{e_2}{e_1}\left(\frac{b_2}{b_1}\right)^2\left(1-2e_1\sigma\frac{b_1}{b_2}\right)\right]\frac{(S_2/S_1)}{\pi A_2 e_2}>0 \quad (10)$$

Therefore, the minimum aircraft drag, or the maximum value of W/D, is given by the tail upload defined by

$$\frac{L_2}{W}\frac{S_1}{S_2}C_{L0}=\frac{L_2}{qS_2}=C_{L_2}=\frac{K_1}{2K_2}>0 \quad (11)$$

Consequently, the minimum aircraft drag is given by

$$C_{D_{\min.}}=C_{D_e}+\frac{C_{L0}{}^2}{\pi A_1 e_1}-\frac{K_1{}^2}{4K_2}=C_{D_e}+\frac{C_{L0}{}^2}{\pi A_1 e_1}(1-K_3) \quad (12)$$

where

$$1>K_3=\frac{(1-e_1\sigma b_1/b_2)^2}{[1-2e_1\sigma b_1/b_2+(e_1/e_2)(b_1/b_2)^2]}>0 \quad (13)$$

as long as $e_1\sigma b_1/b_2<1$ and $b_1/b_2\geq 1$. For most aircraft at minimum drag, $W/40>L_2>0$ and $K_3<0.01$. So the decrease in the induced drag is only 1% of that for zero tail load, corresponding to Eq. (12) with $K_3=0$. However, as shown by the calculations in Table 1, there can be a 3% decrease in total drag if the comparison is made with respect to a typical tail download of $C_{L_2}=-0.11$, when $C_{L0}=0.44$.

If $e_1=1=e_2$, then Eq. (12) reduces to

$$C_{D_{\min.}}=C_{De}+\frac{C_{L0}{}^2}{\pi A_1}\frac{(1-\sigma^2)(b_1/b_2)^2}{(1-2\sigma b_1/b_2+b_1{}^2/b_2{}^2)} \quad (14)$$

The second term corresponds to the minimum induced drag given by Glauert[4] for any biplane. When the gap is zero, corresponding to $\sigma=b_2/b_1$ with the tail in the plane of the wing, then Eq. (14) reduces to the case of $K_3=0$ in Eq. (12), which corresponds to zero tail load, and

$$C_{D(L_2=0)}=C_{De}+\frac{C_{L0}{}^2}{\pi A_1}=C_{De}+\frac{(W/qS_1)^2}{\pi A_1} \quad (15)$$

However, when the tail is either above or below the plane of the wing then $\sigma<b_2/b_1$; Eqs. (5) and (8) state for $L_2>0$ we have less drag, so $C_{D_{\min.}}<C_{D(L_2=0)}$. The physical explanation why a tail upload produces less induced drag than a zero tail load is due to the reduced induced drag on the wing ($C_{L_1}<C_{L0}$) plus the forward rotation of the wing's lift vector due to the circulation vortex system produced by an upload on a tail that is not in the plane of the wing. These two effects are sufficient to overcome the additional induced drag on the tail (which is usually less efficient than the wing since $A_2e_2<A_1e_1$), and the rearward rotation of the tail lift vector. Equations (5) and (8) prove that the induced drag is always less with a tail upload than it is with a download of the same magnitude, the difference being given by $2K_1C_{L_2}$. However, the minimum induced drag usually occurs for such a small tail upload that it is practically the same as that for the zero tail load case given by Eq. (15). The calculations for a typical aircraft are given in Table 2, and it is seen that the difference between the minimum induced drag and that for zero tail load is less than 1%. However, the induced drag produced by a tail download that is only 1/20 of the gross weight W can increase the induced drag by more than 5%. But if the aircraft $C_{De}\geq 0.0122$, then this drag increment is less than 3% of the total aircraft drag which may be written, following Eq. (8) for the aircraft defined in Table 2, with $C_{L0}=0.44$, as

$$C_D=0.0122+0.0088-0.002C_{L_2}+0.02C_{L_2}{}^2 \quad (16)$$

Table 2 shows that the above magnitudes for K_1 and K_2 are reasonable for a tail that is a vertical distance greater than $b_1/20$ from the plane of the wing. Table 1 compares the zero tail load total drag coefficient with that for various tail loads. It is seen that when the tail download equals 1/12 of the aircraft gross weight W, then the total drag is increased by 5%. This, of course, would lead to a corresponding increase in the fuel consumption. Even a tail download of only $W/20$ still increases the total drag coefficient by 2.4% over the minimum drag produced by a very small tail upload. These calculations point out the possibilities of conserving fuel by increasing the tail volume so as to attain a zero or slightly positive tail load.

Naylor's[3] first graph can be obtained from Eq. (5) if the middle term is omitted, since he assumed that $\sigma=b_2/b_1$. Naylor graphed D_i vs L_2/L_1, so his curves are not symmetric about the minimum D_i value at $L_2=0$. If one graphed D_i vs L_2/W, then the curves would be symmetric parabolas about $L_2=0$, since Eq. (5) shows that the induced drag is the same

Table 1 Variation of C_D = Eq. (16) with C_{L_2}

C_{L_2}	C_D	$C_D-C_{D_{\min.}}$	L_2
0	0.021	0.00005	0
0.05	0.02095 (min.)	...	$W/44$
−0.05	0.02115	0.00020	$-W/44$
0.11	0.02102	0.00007	$W/20$
−0.11	0.02146	0.00051	$-W/20$
−0.179	0.022	0.00105	$-W/12.3$
0.279	0.022	0.00105	$W/7.9$

Table 2 Variation of C_{D_i} with σ, e_1, and e_2, when $C_{L0}=0.44$

$C_D-C_{D_e}=C_{D_i}=C_{L0}^2/\pi A_1-K_1C_{L_2}+K_2C_{L_2}^2$ $\quad C_{L0}^2/\pi A_1=0.0088$

$A_2/A_1=0.45$ $\quad b_2/b_1=0.3$ $\quad S_2/S_1=0.2$ $\quad \pi A_1=22$ $\quad \pi A_2=9.9$

e_1	e_2	10^5K_1	10^4K_2	$10^4C_{L_2}$	$10^5C_{D_i}$ min.	$10^5C_{D_i}$ ($L_2=-W/20$)	ΔC_{D_i}
			Gap $=b_1/10$	$\sigma=0.235$	$\sigma b_2/b_1=0.78$		
1.0	1.0	176	192	459	875	922	0.00047
1.0	0.9	176	214	411	876	925	0.00049
0.9	0.9	265	216	612	872	935	0.00063
			Gap $=b_1/20$	$\sigma=0.264$	$\sigma b_2/b_1=0.88$		
1.0	1.0	96	188	255	878	913	0.00035
1.0	0.9	96	211	228	879	916	0.00037
0.9	0.9	185	213	435	876	926	0.00050
			Gap = 0	$\sigma=0.3$	$\sigma b_2/b_1=1$		
1.0	1.0	0	184	0	880	902	0.00022
0.9	0.9	89	208	213	879	915	0.00036

for either upload or download on the tail. However, if $\sigma < b_2/b_l < 1$, then Eqs. (5) and (8) show that the induced drag is always less with a tail upload. However if we rewrote Eq. (8) in terms of C_{L_l}, we could not easily verify this simple result.

It is important to note that the term $\sigma(b_l/b_2)$ $(2C_{L_l}/\pi A_l)$ in Eq. (3) corresponds to the wing downwash in the Trefftz plane, so it is only concerned with that portion of the wing's downwash that is produced by the trailing circulation vortex system. In other words, the tail thrust when $L_2 < 0$ in Eq. (3) is the limiting case as the stagger becomes infinite in ideal potential flow. The factor $\sigma(b_l/b_2) \leq 1$ gives the decrease in the tail thrust as the tail moves up or down from the horizontal plane of the wing. By Munk's stagger theorem, the total induced drag would remain the same for any stagger distance (including zero stagger if the tail is not in the same horizontal plane as the wing). Therefore, for this ideal potential flow case we have the minimum total induced drag given by Eq. (6) for $(L_2/w) \geq 0$. However, it is now common practice to replace this Trefftz-plane wing downwash by the expression $\epsilon_\alpha(\alpha - \alpha_0) + \epsilon_0$, which includes the downwash produced at the tail by both the wing and the fuselage. The errors produced by this substitution, and the effects produced by viscosity and the rolling up of the wings vortex sheet, will be discussed in a subsequent paper. Lutze[5] has shown that, if ϵ_0 is made sufficiently large, then the minimum trim drag will occur with a tail download.

Acknowledgment

The writer would like to dedicate this paper to honor the 65th birthday of J. Weissinger on May 12, 1978.

References

[1] Karman, Th. von and Burgers, J. M., *Durand's Aerodynamic Theory*, Vol. 2, Div. E, J. Springer, Berlin, 1934, pp. 218-220 and 326-329.

[2] Kerber, L. V., *Durand's Aerodynamic Theory*, Vol. 5, Div. O, J. Springer, Berlin, 1934, pp. 333-337.

[3] Naylor, C. H., "Notes on the Induced Drag of Wing-Tail Combination," British R&M 2528, July 1946.

[4] Glauert, H., *The Elements of Aerofoil and Aircrew Theory*, 2nd Ed., Cambridge University Press, 1947, p. 184.

[5] Lutze, F. H., "Trimmed Drag Considerations," *Journal of Aircraft*, Vol. 14, June 1977, pp. 544-546.